

# Dynamic Symmetry Approach to Entanglement

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**Abstract.** In this lectures I explain a connection between geometric invariant theory and entanglement, and give a number of examples how this approach works.

**Keywords.** Entanglement, Dynamic symmetry, Geometric invariant theory

## 1. Physical background

### 1.1. Classical mechanics

Let me start with classical nonlinear equation

$$\frac{d^2\theta}{dt^2} = -\omega^2 \sin \theta, \quad \omega^2 = \frac{g}{\ell} \quad (1)$$

describing graceful swing of a clock pendulum in a corner of Victorian drawing room. It has double periodic solution

$$\theta(t) = \theta(t + T) = \theta(t + i\tau),$$

with real period  $T$ , and imaginary one  $i\tau$ . Out of this equation, carefully studied by Legendre, Abel, and Jacobi, stems the whole theory of elliptic functions.

Physicists are less interested in mathematical subtleties, and usually shrink equation (1) to linear one

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta, \quad |\theta| \ll 1$$

with simple harmonic solution  $\theta = e^{\pm i\omega t}$ . This example outlines a general feature of classical mechanics, where linearity appears mainly as a useful approximation.

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### 1.2. Quantum mechanics

In striking contrast to this, *quantum mechanics* is intrinsically linear, and therefore *more simple* than classical one, in the same way as analytic geometry of Descartes is simpler than synthetic geometry of Euclid. As a price for its simplicity quantum mechanics runs into enormous difficulties to manifest itself in a harsh macroscopic reality. This is what makes quantum phenomenology so tricky.

Mathematicians encounter a similar problem when try to extract geometrical gist from a mess of coordinate calculations. In both cases the challenge is to cover formal bonds of mathematical skeleton with flesh of meaning.

As we know from Klein's *Erlangen program*, the geometrical meaning rests upon invariant quantities and properties (w.r. to a relevant *structure group*  $G$ ). This thesis effectively reduces "elementary" geometry to invariant theory.

As far as physics is concerned, we witnessed its progressive geometrization in the last decades [65,25]. To name few examples: general relativity, gauge theories, from electro-weak interactions to chromodynamics, are all geometrical in their ideal essence. In this lectures, mostly based on preprint [32], I explain a connection between geometric invariant theory and entanglement, and give a number of examples how this approach works. One can find further applications in [33,34].

### 1.3. Von Neumann picture

A background of a quantum system  $A$  is Hilbert space  $\mathcal{H}_A$ , called *state space*. Here, by default, the systems are expected to be *finite*:  $\dim \mathcal{H}_A < \infty$ . A *pure state* of the system is given by unit vector  $\psi \in \mathcal{H}_A$ , or by projector operator  $|\psi\rangle\langle\psi|$ , if the phase factor is irrelevant. Classical mixture  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$  of pure states called *mixed state* or *density matrix*. This is a nonnegative Hermitian operator  $\rho : \mathcal{H}_A \rightarrow \mathcal{H}_A$  with unit trace  $\text{Tr } \rho = 1$ .

An *observable* of the system  $A$  is Hermitian operator  $X_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$ . Actual measurement of  $X_A$  upon the system in state  $\rho$  produces a random quantity  $x_A \in \text{Spec } X_A$  implicitly determined by expectations

$$\langle f(x_A) \rangle_\rho = \text{Tr}(\rho f(X_A)) = \langle \psi | f(X_A) | \psi \rangle$$

for arbitrary function  $f(x)$  on  $\text{Spec } X_A$  (the second equation holds for pure state  $\psi$ ). The measurement process puts the system into an eigenstate  $\psi_\lambda$  with the observed eigenvalue  $\lambda \in \text{Spec } X_A$ . Occasionally we use ambiguous notation  $|\lambda\rangle$  for the eigenstate with eigenvalue  $\lambda$ .

### 1.4. Superposition principle

The linearity of quantum mechanics is embedded from the outset in *Schrödinger equation* describing time evolution of the system

$$i\hbar \frac{d\psi}{dt} = H_A \psi \tag{2}$$

where  $H_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$  is the *Hamiltonian* of the system  $A$ . Being linear Schrödinger equation admits simple solution

$$\psi(t) = U(t)\psi(0), \quad (3)$$

where  $U(t) = \exp\left(-\frac{i}{\hbar} \int_0^t H_A(t) dt\right)$  is unitary *evolution operator*.

Solutions of Schrödinger equation (2) form a linear space. This observation is a source of general *superposition principle*, which claims that a normalized linear combination

$$a\psi + b\varphi$$

of realizable physical states  $\psi, \varphi$  is again a realizable physical state (with no recipe how to cook it). This may be the most important revelation about physical reality after atomic hypothesis. It is extremely counterintuitive and implies, for example, that one can set the celebrated Shcrödinger cat into the state

$$\psi = |\text{dead}\rangle + |\text{alive}\rangle$$

intermediate between death and life. As BBC put it: “*In quantum mechanics it is not so easy to be or not to be.*”

From the superposition principle it follows that state space of composite system  $AB$  splits into *tensor product*

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

of state spaces of the components, as opposed to *direct product*  $P_{AB} = P_A \times P_B$  of configuration spaces in classical mechanics.

### 1.5. Consequences of linearity

The linearity imposes severe restrictions on possible manipulations with quantum states. Here is a couple of examples.

#### 1.5.1. No-cloning Theorem

Let's start with notorious claim

**Theorem** ([67], [12]). *An unknown quantum state can't be duplicated.*

Indeed the cloning process would be given by operator

$$\psi \otimes (\text{state of the Cloning Machine}) \mapsto \psi \otimes \psi \otimes (\text{another state of the Machine})$$

which is *quadratic* in state vector  $\psi$  of the quantum system.  $\square$

#### 1.5.2. Inaccessibility of quantum information

As another application of linearity consider the following

**Theorem.** *No information on quantum system can be gained without destruction of its state.*

Indeed the measurement process is described by linear operator

$$U : \psi_{\text{ini}} \otimes \Psi_{\text{ini}} \mapsto \psi_{\text{fin}} \otimes \Psi_{\text{fin}},$$

where  $\psi$  and  $\Psi$  are states of the system and the measurement device respectively. The initial state  $\Psi_{\text{ini}}$  of the apparatus supposed to be fixed once and for all, so that the final state  $\psi_{\text{fin}} \otimes \Psi_{\text{fin}}$  is a linear function of  $\psi_{\text{ini}}$ . This is possible only if

- $\psi_{\text{fin}}$  is linear in  $\psi_{\text{ini}}$  and  $\Psi_{\text{fin}}$  is independent of  $\psi_{\text{ini}}$ ,
- or vice versa  $\Psi_{\text{fin}}$  is linear in  $\psi_{\text{ini}}$  and  $\psi_{\text{fin}}$  is independent of  $\psi_{\text{ini}}$ .

In the former case the final state of the measurement device contains no information on the system, while in the latter the unknown initial state  $\psi_{\text{ini}}$  is completely erased in the measurement process.  $\square$

Emmanuel Kant, who persistently defended absolute reality of unobservable “thing-in-itself”, or *noumenon*, as opposed to *phenomenon*, should be very pleased with this theorem identifying noumenon with quantum state.

The theorem suggests that complete separation of a system from a measuring apparatus is unlikely. As a rule the system remains *entangled*, with the measuring device, with two exceptions described above.

### 1.6. Reduced states and first glimpse of entanglement

Density matrix of composite system  $AB$  can be written as a linear combination of separable states

$$\rho_{AB} = \sum_{\alpha} a_{\alpha} \rho_A^{\alpha} \otimes \rho_B^{\alpha}, \quad (4)$$

where  $\rho_A^{\alpha}, \rho_B^{\alpha}$  are mixed states of the components  $A, B$  respectively, and the coefficients  $a_{\alpha}$  are *not* necessarily positive. Its *reduced matrices* or *marginal states* may be defined by equations

$$\begin{aligned} \rho_A &= \sum_{\alpha} a_{\alpha} \text{Tr}(\rho_B^{\alpha}) \rho_A^{\alpha} := \text{Tr}_B(\rho_{AB}), \\ \rho_B &= \sum_{\alpha} a_{\alpha} \text{Tr}(\rho_A^{\alpha}) \rho_B^{\alpha} := \text{Tr}_A(\rho_{AB}). \end{aligned}$$

The reduced states  $\rho_A, \rho_B$  are independent of the decomposition (4) and can be characterized intrinsically by the following property

$$\langle X_A \rangle_{\rho_{AB}} = \text{Tr}(\rho_{AB} X_A) = \text{Tr}(\rho_A X_A) = \langle X_A \rangle_{\rho_A}, \quad \forall \quad X_A : \mathcal{H}_A \rightarrow \mathcal{H}_A, \quad (5)$$

which tells that  $\rho_A$  is a “visible” state of subsystem  $A$ . This justifies the terminology.

**Example 1.6.1.** Let’s identify pure state of two component system

$$\psi = \sum_{ij} \psi_{ij} \alpha_i \otimes \beta_j \in \mathcal{H}_A \otimes \mathcal{H}_B$$

with its matrix  $[\psi_{ij}]$  in orthonormal bases  $\alpha_i, \beta_j$  of  $\mathcal{H}_A, \mathcal{H}_B$ . Then the reduced states of  $\psi$  in respective bases are given by matrices

$$\rho_A = \psi^\dagger \psi, \quad \rho_B = \psi \psi^\dagger, \quad (6)$$

which have the same non negative spectra

$$\text{Spec } \rho_A = \text{Spec } \rho_B = \lambda \quad (7)$$

except extra zeros if  $\dim \mathcal{H}_A \neq \dim \mathcal{H}_B$ . The isospectrality implies so called *Schmidt decomposition*

$$\psi = \sum_i \sqrt{\lambda_i} \psi_i^A \otimes \psi_i^B, \quad (8)$$

where  $\psi_i^A, \psi_i^B$  are eigenvectors of  $\rho_A, \rho_B$  with the same eigenvalue  $\lambda_i$ .

In striking contrast to the classical case marginals of a pure state  $\psi \neq \psi_A \otimes \psi_B$  are mixed ones, i.e. as Srödinger put it “*maximal knowledge of the whole does not necessarily includes the maximal knowledge of its parts*” [58]. He coined the term *entanglement* just to describe this phenomenon. Von Neumann entropy of the marginal states provides a natural measure of entanglement

$$E(\psi) = -\text{Tr}(\rho_A \log \rho_A) = -\text{Tr}(\rho_B \log \rho_B) = -\sum_i \lambda_i \log \lambda_i. \quad (9)$$

In equidimensional system  $\dim \mathcal{H}_A = \dim \mathcal{H}_B = n$  maximum of entanglement, equal to  $\log n$  entangled bits (ebits), is attained for a state with scalar reduced matrices  $\rho_A, \rho_B$ .

### 1.7. Quantum dynamical systems

In the above discussion we tacitly suppose, following von Neumann, that all observable  $X_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$  or what is the same all unitary transformations  $e^{itX_A} : \mathcal{H}_A \rightarrow \mathcal{H}_A$  are equally accessible for manipulation with quantum states. However physical nature of the system may impose unavoidable constraints.

**Example 1.7.1.** The components of composite system  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  may be spatially separated by tens of kilometers, as in EPR pairs used in quantum cryptography. In such circumstances only local observations  $X_A$  and  $X_B$  are available. This may be even more compelling if the components are spacelike separated at the moment of measurement.

**Example 1.7.2.** Consider a system of  $N$  identical particles, each with space of internal degrees of freedom  $\mathcal{H}$ . By Pauli principle state space of such system shrinks to *symmetric tensors*  $S^N \mathcal{H} \subset \mathcal{H}^{\otimes N}$  for bosons, and to *skew symmetric tensors*  $\wedge^N \mathcal{H} \subset \mathcal{H}^{\otimes N}$  for fermions. This superselection rule imposes severe restriction on manipulation with quantum states, effectively reducing the accessible measurements to that of a single particle.

**Example 1.7.3.** State space  $\mathcal{H}_s$  of spin  $s$  system has dimension  $2s + 1$ . Measurements upon such system are usually confined to spin projection onto a chosen direction. They generate Lie algebra  $\mathfrak{su}(2)$  rather than full algebra of traceless operators  $\mathfrak{su}(2s + 1)$ .

This consideration led many researchers to the conclusion, that available observables should be included in description of any quantum system from the outset [24,16]. Robert Hermann stated this thesis as follows:

“The basic principles of quantum mechanics seem to require the postulation of a Lie algebra of observables and a representation of this algebra by skew-Hermitian operators.”

We'll refer to the Lie algebra  $\mathfrak{L}$  as *algebra of observables* and to the corresponding group  $G = \exp(i\mathfrak{L})$  as *dynamical symmetry group* of the quantum system in question. Its state space  $\mathcal{H}$  together with unitary representation of the dynamical group  $G : \mathcal{H}$  is said to be *quantum dynamical system*. In contrast to R. Hermann we treat  $\mathfrak{L}$  as algebra of *Hermitian*, rather than by skew-Hermitian operators, and include imaginary unit  $i$  in the definition of Lie bracket  $[X, Y] = i(XY - YX)$ .

The choice of the algebra  $\mathfrak{L}$  depends on the *measurements* we are able to perform over the system, or what is the same the *Hamiltonians* which are accessible for manipulation with quantum states.

For example, if we are restricted to *local measurements* of a system consisting of two remote components  $A, B$  with full access to the local degrees of freedom then the dynamical group is  $SU(\mathcal{H}_A) \times SU(\mathcal{H}_B)$  acting in  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ .

In settings of Example 1.7.2 suppose that a single particle is described by dynamical system  $G : \mathcal{H}$ . Then ensemble of  $N$  identical particles corresponds to dynamical system  $G : S^N \mathcal{H}$  for bosons, and to  $G : \wedge^N \mathcal{H}$  for fermions.

The dynamic group of spin system from Example 1.7.3 is  $SU(2)$  in its spin  $s$  representation  $\mathcal{H}_s$ .

## 2. Coherent states

Coherent states, first introduced by Schrödinger [57] in 1926, lapsed into obscurity for decades until Glauber [22] recovered them in 1963 in connection with laser emission. He have to wait more then 40 years to win Nobel Prize in 2005 for three paper published in 1963-64.

Later in 70th Perelomov [47,48] puts coherent states into general framework of dynamic symmetry groups. We'll use a similar approach for entanglement, and to warm up recall here some basic facts about coherent states.

### 2.1. Glauber coherent states

Let's start with *quantum oscillator*, described by canonical pair of operators  $p, q$ ,  $[p, q] = i\hbar$ , generating *Weyl-Heisenberg algebra*  $\mathcal{W}$ . This algebra has unique unitary irreducible representation, which can be realized in *Fock space*  $\mathcal{F}$  spanned by orthonormal set of  $n$ -excitations states  $|n\rangle$  on which dimensionless annihilation and creation operators

$$a = \frac{q + ip}{\sqrt{2\hbar}}, \quad a^\dagger = \frac{q - ip}{\sqrt{2\hbar}}, \quad [a, a^\dagger] = 1$$

act by formulae

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

A typical element from *Weyl-Heisenberg group*  $W = \exp \mathcal{W}$ , up to a phase factor, is of the form  $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$  for some  $\alpha \in \mathbb{C}$ . Action of this operator on vacuum  $|0\rangle$  produces state

$$|\alpha\rangle := D(\alpha)|0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n \geq 0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (10)$$

known as *Glauber coherent state*. The number of excitations in this state has Poisson distribution with parameter  $|\alpha|^2$ . In many respects its behavior is close to classical, e.g. Heisenberg's uncertainty  $\Delta p \Delta q = \hbar/2$  for this state is minimal possible. In coordinate representation

$$q = x, \quad p = i\hbar \frac{d}{dx}$$

its time evolution is given by harmonic oscillation of Gaussian distribution of width  $\sqrt{\hbar}$  with amplitude  $|\alpha|\sqrt{2\hbar}$ . Therefor for big number of photons  $|\alpha|^2 \gg 1$  coherent states behave classically. Recall also Glauber's theorem [23] which claims that classical field or force excites quantum oscillator into a coherent state.

We'll return to these aspects of coherent states later, and focus now on their mathematical description

*Glauber coherent states = W-orbit of vacuum*

which sounds more suggestive then explicit equation (10).

## 2.2. General coherent states

Let's now turn to arbitrary quantum system  $A$  with dynamical symmetry group  $G = \exp i\mathcal{L}$ . By definition its Lie algebra  $\mathcal{L} = \text{Lie } G$  is generated by all *essential observables* of the system (like  $p, q$  in the above example). To simplify the underling mathematics suppose in addition that state space of the system  $\mathcal{H}_A$  is finite, and representation of  $G$  in  $\mathcal{H}_A$  is irreducible.

To extend (10) to this general setting we have to understand the special role of the vacuum, which primary considered as a *ground state* of a system. For group-theoretical approach, however, another its property is more relevant:

*Vacuum is a state with maximal symmetry.*

This may be also spelled out that vacuum is a most degenerate state of a system.

## 2.3. Complexified dynamical group

Symmetries of state  $\psi$  are given by its *stabilizers*

$$G_\psi = \{g \in G \mid g\psi = \mu(g)\psi\}, \quad \mathcal{L}_\psi = \{X \in \mathcal{L} \mid X\psi = \lambda(X)\psi\} \quad (11)$$

in the dynamical group  $G$  or in its Lie algebra  $\mathcal{L} = \text{Lie } G$ . Here  $\mu(g)$  and  $\lambda(X)$  are scalars. Looking back to the quantum oscillator, we see that some symmetries are ac-

tually hidden, and manifest themselves only in *complexified* algebra  $\mathfrak{L}^c = \mathfrak{L} \otimes \mathbb{C}$  and group  $G^c = \exp \mathfrak{L}^c$ . For example, stabilizer of vacuum  $|0\rangle$  in Weyl-Heisenberg algebra  $\mathcal{W}$  is trivial  $\mathcal{W}_{|0\rangle} = \text{scalars}$ , while in complexified algebra  $\mathcal{W}^c$  it contains annihilation operator,  $\mathcal{W}_{|0\rangle}^c = \mathbb{C} + \mathbb{C}a$ . In the last case the stabilizer is big enough to recover the whole dynamical algebra

$$\mathcal{W}^c = \mathcal{W}_{|0\rangle}^c + \mathcal{W}_{|0\rangle}^{c\dagger}.$$

This decomposition, called *complex polarization*, gives a precise meaning for the maximal degeneracy of a vacuum or a coherent state.

#### 2.4. General definition of coherent state

State  $\psi \in \mathcal{H}$  is said to be *coherent* if

$$\mathfrak{L}^c = \mathfrak{L}_\psi^c + \mathfrak{L}_\psi^{c\dagger}$$

In finite dimensional case all such decompositions come from *Borel subalgebra*, i.e. a maximal solvable subalgebra  $\mathfrak{B} \subset \mathfrak{L}^c$ . The corresponding *Borel subgroup*  $B = \exp \mathfrak{B}$  is a minimal subgroup of  $G^c$  with compact factor  $G^c/B$ . Typical example is subgroup of upper triangular matrices in  $\text{SL}(n, \mathbb{C}) = \text{complexification of } \text{SU}(n)$ . It is a basic structural fact that  $\mathfrak{B} + \mathfrak{B}^\dagger = \mathfrak{L}^c$ , and therefore

$$\psi \text{ is coherent} \Leftrightarrow \psi \text{ is an eigenvector of } \mathfrak{B}$$

In representation theory eigenstate  $\psi$  of  $\mathfrak{B}$  is called *highest vector*, and the corresponding eigenvalue  $\lambda = \lambda(X)$ ,

$$X\psi = \lambda(X)\psi, \quad X \in \mathfrak{B}$$

is said to be *highest weight*.

Here are the basic properties of coherent states.

- For irreducible system  $G : \mathcal{H}$  the highest vector  $\psi_0$  (=vacuum) is unique.
- There is only one irreducible representation  $\mathcal{H} = \mathcal{H}_\lambda$  with highest weight  $\lambda$ .
- All coherent states are of the form  $\psi = g\psi_0$ ,  $g \in G$ .
- Coherent state  $\psi$  in composite system  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  with dynamical group  $G_{AB} = G_A \times G_B$  splits into product  $\psi = \psi_1 \otimes \psi_2$  of coherent states of the components.

*Remark.* Coherent state theory, in the form given by Perelomov [48], is a physical equivalent of Kirillov–Kostant *orbit method* [31] in representation theory.

The complexified group play crucial role in our study. Its operational interpretation may vary. Here is a couple of examples.



**Example 2.4.1.** *Spin systems.* For system of spin  $s$  (see example 1.7.3) coherent states have definite spin projection  $s$  onto some direction

$$\psi \text{ is coherent} \iff \psi = |s\rangle.$$

Complexification of spin group  $SU(2)$  is group of unimodular matrices  $SL(2, \mathbb{C})$ . The latter is locally isomorphic to *Lorentz group* and controls relativistic transformation of spin states in a moving frame.

**Example 2.4.2.** For two component system  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  with full access to local degrees of freedom the coherent states are decomposable ones

$$\psi_{AB} \text{ is coherent} \iff \psi_{AB} = \psi_A \otimes \psi_B.$$

The dynamical group of this system is  $G = SU(\mathcal{H}_A) \times SU(\mathcal{H}_B)$ , see example 1.7.1. Its complexification  $G^c = SL(\mathcal{H}_A) \times SL(\mathcal{H}_B)$  has an important quantum informational interpretation as group of invertible Stochastic Local Operations assisted with Classical Communication (SLOCC transformations), see [61]. These are essentially LOCC operations with postselection.

## 2.5. Total variance

Let's define *total variance* of state  $\psi$  by equation

$$\mathbb{D}(\psi) = \sum_i \langle \psi | X_i^2 | \psi \rangle - \langle \psi | X_i | \psi \rangle^2 \quad (12)$$

where  $X_i \in \mathfrak{L}$  form an orthonormal basis of the Lie algebra of essential observables with respect to its invariant metric (for spin group  $SU(2)$  one can take for the basis spin projector operators  $J_x, J_y, J_z$ ). The total variance is independent of the basis  $X_i$ , hence  $G$ -invariant. It measures the total level of *quantum fluctuations* of the system in state  $\psi$ .

The first sum in (12) contains well known *Casimir operator*

$$C = \sum_i X_i^2$$

which commutes with  $G$  and hence acts as a scalar in every irreducible representation. Specifically

**Theorem 2.5.1.** *The Casimir operator  $C$  acts in irreducible representation  $\mathcal{H}_\lambda$  of highest weight  $\lambda$  as multiplication by scalar  $C_\lambda = \langle \lambda, \lambda + 2\delta \rangle$ .*

One can use two *dual bases*  $X_i$  and  $X^j$  of  $\mathfrak{L}$ , with respect to invariant bilinear form  $B(X_i, X^j) = \delta_{ij}$  to construct the Casimir operator

$$C = \sum_i X_i X^i.$$

For example, take basis of  $\mathfrak{L}$  consisting of orthonormal basis  $H_i$  of Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{L}$  and its *root vectors*  $X_\alpha \in \mathfrak{L}$  normalized by condition  $B(X_\alpha, X_{-\alpha}) = 1$ . Then the dual basis is obtained by substitution  $X_\alpha \mapsto X_{-\alpha}$  and hence

$$C = \sum_i H_i^2 + \sum_{\alpha=\text{root}} X_\alpha X_{-\alpha} = \sum_i H_i^2 + \sum_{\alpha>0} H_\alpha + 2 \sum_{\alpha>0} X_{-\alpha} X_\alpha,$$

where in the last equation we use commutation relation  $[X_\alpha, X_{-\alpha}] = H_\alpha$ . Applying this to the highest vector  $\psi \in \mathcal{H}$  of weight  $\lambda$ , which by definition is annihilated by all operators  $X_\alpha$ ,  $\alpha > 0$  and  $H\psi = \lambda(H)\psi$ ,  $H \in \mathfrak{h}$ , we get

$$C\psi = \sum_i \lambda(H_i)^2 \psi + \sum_{\alpha>0} \lambda(H_\alpha) \psi = \langle \lambda, \lambda + 2\delta \rangle \psi, \quad (13)$$

where  $2\delta = \sum_{\alpha>0} \alpha$  is the sum of positive roots and  $\langle *, * \rangle$  is the invariant form  $B$  translated to the dual space  $\mathfrak{h}^*$ . Hence Casimir operator  $C$  acts as scalar  $C_\lambda = \langle \lambda, \lambda + 2\delta \rangle$  in irreducible representation with highest weight  $\lambda$ .  $\square$

### 2.6. Extremal property of coherent states

For spin  $s$  representation  $\mathcal{H}_s$  of  $\text{SU}(2)$  the Casimir is equal to square of the total moment

$$C = J^2 = J_x^2 + J_y^2 + J_z^2 = s(s+1).$$

Hence

$$\mathbb{D}(\psi) = \langle \lambda, \lambda + 2\delta \rangle - \sum_i \langle \psi | X_i | \psi \rangle^2. \quad (14)$$

**Theorem 2.6.1** (Delbourgo and Fox [11]). *State  $\psi$  is coherent iff its total variance is minimal possible, and in this case*

$$\mathbb{D}(\psi) = \langle \lambda, 2\delta \rangle.$$

Let  $\rho = |\psi\rangle\langle\psi|$  be pure state and  $\rho_{\mathfrak{L}}$  be its orthogonal projection into subalgebra  $\mathfrak{L} \subset \text{Herm}(\mathcal{H})$  of algebra of all Hermitian operators in  $\mathcal{H}$  with trace metric  $(X, Y) = \text{Tr}(X \cdot Y)$ . By definition we have

$$\langle \psi | X | \psi \rangle = \text{Tr}_{\mathcal{H}}(\rho X) = \text{Tr}_{\mathcal{H}}(\rho_{\mathfrak{L}} X), \quad \forall X \in \mathfrak{L}.$$

Choose a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{L}$  containing  $\rho_{\mathfrak{L}}$ . Then  $\langle \psi | X_i | \psi \rangle = \text{Tr}_{\mathcal{H}}(\rho_{\mathfrak{L}} X_i) = 0$  for  $X_i \perp \mathfrak{h}$  and we can restrict the sum in (14) to orthonormal basis  $H_i$  of Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{L}$  for which by the definition of highest weight  $\langle \psi | H | \psi \rangle^2 \leq \lambda(H)^2$  with equality for the highest vector  $\psi$  only. Hence

$$\sum_i \langle \psi | X_i | \psi \rangle^2 = \sum_i \langle \psi | H_i | \psi \rangle^2 \leq \sum_i \lambda(H_i)^2 = \langle \lambda, \lambda \rangle, \quad (15)$$

and therefore  $\mathbb{D}(\psi) \geq \langle \lambda, \lambda + 2\delta \rangle - \langle \lambda, \lambda \rangle = \langle \lambda, 2\delta \rangle$ , with equality for coherent states only.  $\square$

The theorem supports the thesis that coherent states are closest to classical ones, cf.  $n^\circ$  2.1. Note however that such simple characterization holds only for finite dimensional systems. The total variance, for example, makes no sense for quantum oscillator, for which we have *minimal uncertainty*  $\Delta p \Delta q = \hbar/2$  instead.

**Example 2.6.1.** For coherent state of spin  $s$  system Theorem 2.6.1 gives  $\mathbb{D}(\psi) = s$ . Hence amplitude of quantum fluctuations  $\sqrt{s}$  for such state is of smaller order then spin  $s$ , which by Example 2.4.1 has a definite direction. Therefor for  $s \rightarrow \infty$  such state looks like a classical rigid body rotating around the spin axis.

### 2.7. Quadratic equations defining coherent states

There is another useful description of coherent states by a system of quadratic equations.

**Example 2.7.1.** Consider two component system  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  with full access to local degrees of freedom  $G = \text{SU}(\mathcal{H}_A) \otimes \text{SU}(\mathcal{H}_B)$ . Coherent states in this case are just separable states  $\psi = \psi_A \otimes \psi_B$  with density matrix  $\rho = |\psi\rangle\langle\psi|$  of rank one. Such matrices can be characterized by vanishing of all minors of order two. Hence coherent states of two component system can be described by a system of quadratic equations.

It turns out that a similar description holds for arbitrary irreducible system  $G : \mathcal{H}_\lambda$  with highest weight  $\lambda$ , see [37].

**Theorem 2.7.1.** State  $\psi \in \mathcal{H}_\lambda$  is coherent iff  $\psi \otimes \psi$  is eigenvector of the Casimir operator  $C$  with eigenvalue  $\langle 2\lambda + 2\delta, 2\lambda \rangle$

$$C(\psi \otimes \psi) = \langle 2\lambda + 2\delta, 2\lambda \rangle (\psi \otimes \psi). \quad (16)$$

Indeed, if  $\psi$  is highest vector of weight  $\lambda$  then  $\psi \otimes \psi$  is a highest vector of weight  $2\lambda$  and equation (16) follows from (13).

Vice versa, in terms of orthonormal basis  $X_i$  of Lie algebra  $\mathfrak{L} = \text{Lie } G$  the Casimir operator in the doublet  $\mathcal{H}_\lambda \otimes \mathcal{H}_\lambda$  looks as follows

$$C = \sum_i (X_i \otimes 1 + 1 \otimes X_i)^2 = \sum_i X_i^2 \otimes 1 + 1 \otimes X_i^2 + 2 \sum_i X_i \otimes X_i.$$

Hence under conditions of the theorem

$$\langle 2\lambda + 2\delta, 2\lambda \rangle = \langle \psi \otimes \psi | C | \psi \otimes \psi \rangle = 2\langle \lambda + \delta, \lambda \rangle + 2 \sum_i \langle \psi | X_i | \psi \rangle^2.$$

It follows that

$$\sum_i \langle \psi | X_i | \psi \rangle^2 = \langle \lambda, \lambda \rangle$$

and hence by inequality (15) state  $\psi$  is coherent.  $\square$

**2.7.2 Remark.** The above calculation show that equation (16) is equivalent to

$$\sum_i X_i \psi \otimes X_i \psi = \langle \lambda, \lambda \rangle \psi \otimes \psi, \quad (17)$$

which in turn amounts to a system of *quadratic equations* on the components of a coherent state  $\psi$ .

**Example 2.7.2.** For spin  $s$  system the theorem tells that state  $\psi$  is coherent iff  $\psi \otimes \psi$  has definite spin  $2s$ . Equations (17) amounts to

$$J_x \psi \otimes J_x \psi + J_y \psi \otimes J_y \psi + J_z \psi \otimes J_z \psi = s^2 \psi \otimes \psi.$$

## 3. Entanglement

From a thought experiment for testing the very basic principles of quantum mechanics in its earlier years [15,58] entanglement nowadays is growing into an important tool

for quantum information processing. Surprisingly enough so far there is no agreement among the experts on the very definition and the origin of entanglement, except unanimous conviction in its fundamental nature and in necessity of its better understanding.

Here we discuss a novel approach to entanglement [32], based on dynamical symmetry group, which puts it into a broader context, eventually applicable to all quantum systems. This sheds new light on known results providing for them a unified conceptual framework, opens a new prospect for further development of the subject, reveals its deep and unexpected connections with other branches of physics and mathematics, and provides an insight on conditions in which entangled state can be stable.

### 3.1. What is entanglement?

Everybody knows, and nobody understand what is entanglement. Here are some virtual answers to the question borrowed from Dagmar Bruß collection [6]:

- J. Bell: ... *a correlation that is stronger then any classical correlation.*
- D. Mermin: ... *a correlation that contradicts the theory of elements of reality.*
- A. Peres: ... *a trick that quantum magicians use to produce phenomena that cannot be imitated by classical magicians.*
- C. Bennet: ... *a resource that enables quantum teleportation.*
- P. Shor: ... *a global structure of wave function that allows the faster algorithms.*
- A. Ekert: ... *a tool for secure communication.*
- Horodecki family: ... *the need for first application of positive maps in physics.*

This list should be enhanced with extensively cited Schrödinger's definition given in *n*° 1.6. The very term was coined by Schrödinger in the famous "cat paradox" paper [58] which in turn was inspired by the no less celebrated Einstein–Podolsky–Rosen *gedanken* experiment [15]. While the latter authors were amazed by nonlocal nature of correlations between the involved particles, J. Bell was the first to note that the correlations themselves, putting aside the nonlocality, are inconsistent with classical realism. Since then Bell's inequalities are produced in industrial quantities and remain the main tool for testing "genuine" entanglement. Note however that in some cases LOCC operations can transform a classical state into nonclassical one [54]. Besides in a sense every quantum system of dimension at least three is nonclassical, see *n*° 3.4 and [40,41].

Below we briefly discuss the nonlocality and violation of classical realism. Neither of this effects, however, allow decisively characterize entangled states. Therefor eventually we turn to another approach, based on the dynamical symmetry group.

### 3.2. EPR paradox

Decay of a spin zero state into two components of spin 1/2 subjects to a strong correlation between spin projections of the components, caused by conservation of moment. The correlation creates an apparent *information channel* between the components, acting beyond their light cones.

Let me emphasize that quantum mechanics refuted the possibility that the spin projection have been fixed at the moment of decay, rather then at the moment of measurement. Otherwise two spatially separated observers can see the same event like burst of a supernova simultaneously even if they are spacelike separated, see [50]. There is no such "event" or "physical reality" in the Bohm version of EPR experiment.

This paradox, recognized in early years of quantum mechanics [15,3], nowadays has many applications, but no intuitive explanation. It is so disturbing that sometimes physicists just ignore it. For example, one of the finest recent book justifies QFT commutation relations as follows [69]:

*A basic relativistic principle states that if two spacetime points are spacelike with respect to each other then no signal can propagate between them, and hence the measurement of an observable at one of the points cannot influence the measurement of another observable at the other point.*

Experiments with EPR pairs tell just the opposite [1,19]. I am not in position to comment this *nonlocality* phenomenon, and therefor turn to less involved *Bell's approach*, limited to the quantum correlations *per se*.

### 3.3. Bell's inequalities

Let's start with *classical marginal problem* which asks for existence of a “body” in  $\mathbb{R}^n$  with given projections onto some coordinate subspaces  $\mathbb{R}^I \subset \mathbb{R}^n$ ,  $I \subset \{1, 2, \dots, n\}$ , i.e. existence of *probability density*  $p(x) = p(x_1, x_2, \dots, x_n)$  with given *marginal distributions*

$$p_I(x_I) = \int_{\mathbb{R}^J} p(x) dx_J, \quad J = \{1, 2, \dots, n\} \setminus I.$$

In discrete version the classical MP amounts to calculation of an image of a multidimensional simplex, say  $\Delta = \{p_{ijk} \geq 0 \mid \sum p_{ijk} = 1\}$ , under a linear map like

$$\pi : \mathbb{R}^{\ell mn} \rightarrow \mathbb{R}^{\ell m} \oplus \mathbb{R}^{mn} \oplus \mathbb{R}^{n\ell},$$

$$p_{ijk} \mapsto (p_{ij}, p_{jk}, p_{ki}),$$

$$p_{ij} = \sum_k p_{ijk}, \quad p_{jk} = \sum_i p_{ijk}, \quad p_{ki} = \sum_j p_{ijk}.$$

The image  $\pi(\Delta)$  is convex hull of  $\pi(\text{Vertices } \Delta)$ . So the classical MP amounts to calculation of facets of a convex hull. In high dimensions this may be a computational nightmare [17,52].

**Example 3.3.1. Classical realism.** Let  $X_i : \mathcal{H}_A \rightarrow \mathcal{H}_A$  be observables of quantum system  $A$ . Actual measurement of  $X_i$  produces random quantity  $x_i$  with values in  $\text{Spec}(X_i)$  and density  $p_i(x_i)$  implicitly determined by expectations

$$\langle f(x_i) \rangle = \langle \psi | f(X_i) | \psi \rangle$$

for all functions  $f$  on spectrum  $\text{Spec}(X_i)$ . For *commuting* observables  $X_i, i \in I$  the random variables  $x_i, i \in I$  have *joint distribution*  $p_I(x_I)$  defined by similar equation

$$\langle f(x_I) \rangle = \langle \psi | f(X_I) | \psi \rangle, \quad \forall f. \quad (18)$$

*Classical realism* postulates existence of a hidden joint distribution of *all variables*  $x_i$ . This amounts to compatibility of the marginal distributions (18) for *commuting* sets of observables  $X_I$ . *Bell inequalities*, designed to test classical realism, stem from the classical marginal problem.

**Example 3.3.2.** Observations of disjoint components of two qubit system  $\mathcal{H}_A \otimes \mathcal{H}_B$  always commute. Let  $A_i, B_j$  be spin projection operators in sites  $A, B$  onto directions  $i, j$ . Their observed values  $a_i, b_j = \pm 1$  satisfy inequality

$$a_1 b_1 + a_2 b_1 + a_2 b_2 - a_1 b_2 + 2 \geq 0.$$

Indeed product of the monomials  $\pm a_i b_j$  in LHS is equal to  $-1$ . Hence one of the monomials is equal to  $+1$  and sum of the rest is  $\geq -3$ .

If all the observables have a hidden joint distribution then taking the expectations we arrive at *Clauser-Horne-Shimony-Holt* inequality for testing “classical realism”

$$\langle \psi | A_1 B_1 | \psi \rangle + \langle \psi | A_2 B_1 | \psi \rangle + \langle \psi | A_2 B_2 | \psi \rangle - \langle \psi | A_1 B_2 | \psi \rangle + 2 \geq 0. \quad (19)$$

All other marginal constraints can be obtained from it by spin flips  $A_i \mapsto \pm A_i$ .

**Example 3.3.3.** For three qubits with two measurements per site the marginal constraints amounts to 53856 independent inequalities, see [53].

Bell’s inequalities make it impossible to model quantum mechanics by classical means. In particular, there is no way to reduce quantum computation to classical one.

### 3.4. Pentagon inequality

Here I’ll give an account of nonclassical states in spin 1 system. Its optical version, called *biphoton*, is a hot topic both for theoretical and experimental studies [59,28,64]. The so-called *neutrally polarised* state of biphoton routinely treated as entangled, since a beam splitter can transform it into a EPR pair of photons. This is the simplest one component system which manifests entanglement.

Spin 1 state space may be identified with complexification of Euclidean space  $\mathbb{E}^3$

$$\mathcal{H} = \mathbb{E}^3 \otimes \mathbb{C},$$

where spin group  $SU(2)$ , locally isomorphic to  $SO(3)$ , acts via rotations of  $\mathbb{E}^3$ . Hilbert space  $\mathcal{H}$  inherits from  $\mathbb{E}^3$  bilinear scalar and cross products, to be denoted by  $(x, y)$  and  $[x, y]$  respectively. Its Hermitian metric is given by  $\langle x | y \rangle = (x^*, y)$  where star means complex conjugation. In this model spin projection operator onto direction  $\ell \in \mathbb{E}^3$  is given by equation

$$J_\ell \psi = i[\ell, \psi].$$

It has real eigenstate  $|0\rangle = \ell$  and two complex conjugate ones  $|\pm 1\rangle = \frac{1}{\sqrt{2}}(m \pm in)$ , where  $(\ell, m, n)$  is orthonormal basis of  $\mathbb{E}^3$ . The latter states are *coherent*, see Example 2.4.1. They may be identified with isotropic vectors

$$\psi \text{ is coherent} \iff (\psi, \psi) = 0.$$

Their properties are drastically different from real vectors  $\ell \in \mathbb{E}^3$  called *completely entangled* spin states. They may be characterized mathematically as follows

$$\psi \text{ is completely entangled} \iff [\psi, \psi^*] = 0.$$

Recall from Example 2.4.1 that Lorentz group, being complexification of  $\text{SO}(3)$ , preserves the bilinear form  $(x, y)$ . Therefore it transforms a coherent state into another coherent state. This however fails for completely entangled states.

Every noncoherent state can be transformed into completely entangled one by a Lorentz boost. In this respect Lorentz group plays rôle similar to SLOCC transform for two qubits which allows to filter out a nonseparable state into a completely entangled Bell state, cf. Example 2.4.2.

By a rotation every spin 1 state can be put into the *canonical form*

$$\psi = m \cos \varphi + in \sin \varphi, \quad 0 \leq \varphi \leq \frac{\pi}{4}. \quad (20)$$

The angle  $\varphi$ , or *generalized concurrence*  $\mu(\psi) = \cos 2\varphi$ , is unique intrinsic parameter of spin 1 state. The extreme values  $\varphi = 0, \pi/4$  correspond to completely entangled and coherent states respectively.

Observe that

$$J_\ell^2 \psi = -[\ell, [\ell, \psi]] = \psi - (\ell, \psi)\ell$$

so that

$$S_\ell = 2J_\ell^2 - 1 : \psi \mapsto \psi - 2(\ell, \psi)\ell$$

is reflection in plane orthogonal to  $\ell$ . Hence  $S_\ell^2 = 1$  and operators  $S_\ell$  and  $S_m$  commute iff  $\ell \perp m$ .

Consider now a cyclic quintuplet of unit vectors  $\ell_i \in \mathbb{E}^3$ ,  $i \bmod 5$ , such that  $\ell_i \perp \ell_{i+1}$ , and call it *pentagram*. Put  $S_i := S_{\ell_i}$ . Then  $[S_i, S_{i+1}] = 0$  and for all possible values  $s_i = \pm 1$  of observable  $S_i$  the following inequality holds

$$s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_5 + s_5 s_1 + 3 \geq 0. \quad (21)$$

Indeed product of the monomials  $s_i s_{i+1}$  is equal to  $+1$ , hence at least one of them is  $+1$ , and the sum of the rest is  $\geq -4$ .

Being commutative, observables  $S_i, S_{i+1}$  have a joint distribution. If all  $S_i$  would have a hidden joint distribution then taking average of (21) one get Bell's type inequality

$$\langle \psi | S_1 S_2 | \psi \rangle + \langle \psi | S_2 S_3 | \psi \rangle + \langle \psi | S_3 S_4 | \psi \rangle + \langle \psi | S_4 S_5 | \psi \rangle + \langle \psi | S_5 S_1 | \psi \rangle + 3 \geq 0 \quad (22)$$

for testing classical realism. Note that all marginal constraints can be obtained from this inequality by flips  $S_i \mapsto \pm S_i$ . Using equation  $S_i = 1 - 2|\ell_i\rangle\langle\ell_i|$  one can recast it into geometrical form

$$\sum_{i \bmod 5} |\langle \ell_i, \psi \rangle|^2 \leq 2 \iff \sum_{i \bmod 5} \cos^2 \alpha_i \leq 2, \quad \alpha_i = \widehat{\ell_i \psi}. \quad (23)$$

Completely entangled spin states easily violate this inequality. Say for regular pentagram and  $\psi \in \mathbb{E}^3$  directed along its axis of symmetry one gets

$$\sum_{i \bmod 5} \cos^2 \alpha_i = \frac{5 \cos \pi/5}{1 + \cos \pi/5} \approx 2.236 > 2.$$

We'll see below that in a smaller extend every non-coherent spin state violates inequality (23) for an appropriate pentagram. The coherent states, on the contrary, pass this test for any pentagram.

To prove these claims write inequality (23) in the form

$$\langle \psi | A | \psi \rangle \leq 2, \quad A = \sum_{i \bmod 5} |\ell_i\rangle\langle \ell_i|,$$

and observe the following properties of spectrum  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$  of operator  $A$ .

1.  $\text{Tr } A = \lambda_1 + \lambda_2 + \lambda_3 = 5$ .
2. If the pentagram contains parallel vectors  $\ell_i \parallel \ell_j$  then  $\lambda_1 = \lambda_2 = 2, \lambda_3 = 1$ .
3. For any pentagram with no parallel vectors
  - (a)  $\lambda_1 > 2$ ,
  - (b)  $\lambda_3 > 1$ ,
  - (c)  $\lambda_2 < 2$ .

*Proof.* (1)  $\text{Tr } A = \sum_{i \bmod 5} \text{Tr } |\ell_i\rangle\langle \ell_i| = 5$ .

(2) Let say  $\ell_1 = \pm \ell_3$  then  $\ell_3, \ell_4, \ell_5$  form orthonormal basis of  $\mathbb{E}^3$ . Hence  $A$  is sum of identical operator  $|\ell_3\rangle\langle \ell_3| + |\ell_4\rangle\langle \ell_4| + |\ell_5\rangle\langle \ell_5|$  and projector  $|\ell_1\rangle\langle \ell_1| + |\ell_2\rangle\langle \ell_2|$  onto plane  $\langle \ell_1, \ell_2 \rangle$ .

(3a) Take unit vector

$$x \in \langle \ell_1, \ell_2 \rangle \cap \langle \ell_3, \ell_4 \rangle$$

so that  $x = \langle \ell_1, x \rangle \ell_1 + \langle \ell_2, x \rangle \ell_2 = \langle \ell_3, x \rangle \ell_3 + \langle \ell_4, x \rangle \ell_4$ . Then

$$Ax = \langle \ell_1, x \rangle \ell_1 + \langle \ell_2, x \rangle \ell_2 + \langle \ell_3, x \rangle \ell_3 + \langle \ell_4, x \rangle \ell_4 + \langle \ell_5, x \rangle \ell_5 = 2x + \langle \ell_5, x \rangle \ell_5$$

and  $\lambda_1 \geq \langle x | A | x \rangle = 2 + |\langle x | \ell_5 \rangle|^2 > 2$ .

(3b) This property is more subtle. It amounts to positivity of the form

$$B(x, y) = \langle x | A - 1 | y \rangle = \sum_{i \bmod 5} \langle x | \ell_i \rangle \langle \ell_i | y \rangle - \langle x | y \rangle.$$

One can show that

$$\det B = 2 \det A \prod_{i < j} \sin^2(\widehat{\ell_i \ell_j}).$$

This implies that  $B$  is nondegenerate for every pentagram of noncollinear vectors. Therefore  $B$  has the same inertia index for all such pentagrams. Finally one can check that for



regular pentagram  $B$  is positive.

(3c) Follows from (1), (3a), and (3b).  $\square$

**Theorem 3.4.1.** *Bell's inequality  $\langle \psi | A | \psi \rangle \leq 2$  holds for coherent state  $\psi$  and any pentagram, while non-coherent state violates this inequality for some pentagram.*

*Proof.* Take  $\psi = m \cos \varphi + in \sin \varphi$ ,  $0 \leq \varphi \leq \pi/4$  in canonical form (20). Then

$$\langle \psi | A | \psi \rangle = \langle m | A | m \rangle \cos^2 \varphi + \langle n | A | n \rangle \sin^2 \varphi.$$

To violate Bell's inequality we have to make the right hand side maximal. This happens for  $m = |\lambda_1\rangle$ , the eigenvector of  $A$  with maximal eigenvalue  $\lambda_1$ , and  $n = |\lambda_2\rangle$ . The maximal value thus obtained is

$$\langle \psi | A | \psi \rangle_{\max} = \lambda_1 \cos^2 \varphi + \lambda_2 \sin^2 \varphi = \frac{\lambda_1 + \lambda_2}{2} + \frac{\lambda_1 - \lambda_2}{2} \cos 2\varphi. \quad (24)$$

For coherent state  $2\varphi = \pi/2$  we arrive at Bell's inequality

$$\langle \psi | A | \psi \rangle_{\max} = \frac{\lambda_1 + \lambda_2}{2} = \frac{5 - \lambda_3}{2} \leq 2$$

which holds for all pentagrams by property (3b). The other part of the theorem follows from the following

**Claim.** *For every noncoherent state  $0 \leq \varphi < \pi/4$  there exists pentagram s.t.*

$$\langle \psi | A | \psi \rangle_{\max} = \frac{\lambda_1 + \lambda_2}{2} + \frac{\lambda_1 - \lambda_2}{2} \cos 2\varphi > 2.$$

Indeed, for degenerate pentagram  $\Pi$ , containing parallel vectors, the corresponding operator  $A$  has multiple eigenvalue  $\lambda_1 = \lambda_2 = 2$  and simple one  $\lambda_3 = 1$ . In this case equation (24) amounts to  $\langle \psi | A | \psi \rangle_{\max} = 2$ . Let  $\tilde{A}$  be operator corresponding to a small nondegenerate  $\varepsilon$ -perturbation  $\tilde{\Pi}$  of pentagram  $\Pi$ , and  $\tilde{\lambda}$  be its spectrum. Then for simple eigenvalue  $\lambda_3$  we have by property (3b)

$$\Delta(\lambda_3) = \tilde{\lambda}_3 - \lambda_3 = O(\varepsilon) > 0,$$

and hence

$$\Delta(\lambda_1 + \lambda_2) = \Delta(5 - \lambda_3) = O(\varepsilon) < 0.$$

Hereafter  $O(\varepsilon)$  denote a quantity of *exact* order  $\varepsilon$ . The increment of multiple roots  $\lambda_1, \lambda_2$  is of smaller order

$$\Delta(\lambda_1) = O(\sqrt{\varepsilon}) > 0, \quad \Delta(\lambda_2) = O(\sqrt{\varepsilon}) < 0, \quad \Delta(\lambda_1 - \lambda_2) = O(\sqrt{\varepsilon}) > 0,$$

where the signs of the increments are derived from properties (3a) and (3c). As result

$$\Delta(\langle \psi | A | \psi \rangle_{\max}) = \Delta\left(\frac{\lambda_1 + \lambda_2}{2} + \frac{\lambda_1 - \lambda_2}{2} \cos 2\varphi\right) = O(\varepsilon) + O(\sqrt{\varepsilon}) = O(\sqrt{\varepsilon}) > 0,$$

provided  $\cos 2\varphi > 0$  and  $\varepsilon \ll 1$ . Hence for noncoherent state Bell's inequality fails:  $\langle \psi | \tilde{A} | \psi \rangle_{\max} > 2$ .  $\square$

**3.4.2 Remark.** Product of orthogonal reflections  $S_i S_{i+1}$  in pentagram inequality (22) is a rotation by angle  $\pi$  in plane  $\langle \ell_i, \ell_{i+1} \rangle$ , i.e.

$$S_i S_{i+1} = 1 - 2J_{[\ell_i, \ell_{i+1}]}^2,$$

and the inequality can be written in the form

$$\langle \psi | J_{[\ell_1, \ell_2]}^2 | \psi \rangle + \langle \psi | J_{[\ell_2, \ell_3]}^2 | \psi \rangle + \langle \psi | J_{[\ell_3, \ell_4]}^2 | \psi \rangle + \langle \psi | J_{[\ell_4, \ell_5]}^2 | \psi \rangle + \langle \psi | J_{[\ell_5, \ell_1]}^2 | \psi \rangle \leq 4.$$

Observe that  $\ell_i, \ell_{i+1}, [\ell_1, \ell_{i+1}]$  are orthogonal and therefore

$$J_{\ell_i}^2 + J_{\ell_{i+1}}^2 + J_{[\ell_i, \ell_{i+1}]}^2 = 2.$$

This allows return to operators  $J_i = J_{\ell_i}$

$$\langle \psi | J_1^2 | \psi \rangle + \langle \psi | J_2^2 | \psi \rangle + \langle \psi | J_3^2 | \psi \rangle + \langle \psi | J_4^2 | \psi \rangle + \langle \psi | J_5^2 | \psi \rangle \geq 3.$$

The last inequality can be tested experimentally by *measuring*  $J$  and *calculating* the average of  $J^2$ . Thus we managed to test classical realism in framework of spin 1 dynamical system in which no two operators  $J \in \mathfrak{su}(2)$  commutes, cf. Example 3.3.1. The trick is that *squares* of the operators may commute.

**3.4.3 Remark.** The difference between coherent and entangled spin states disappears for the full group  $\text{SU}(\mathcal{H})$ . Hence with respect to this group all states are nonclassical, provided  $\dim \mathcal{H} \geq 3$ , cf. [49].

### 3.5. Call for new approach

Putting aside highly publicized philosophical aspects of entanglement, its physical manifestation usually associated with two phenomena:

- *violation of classical realism,*
- *nonlocality.*

As we have seen above every state of a system of dimension  $\geq 3$  with full dynamical group  $\text{SU}(\mathcal{H})$  is nonclassical. Therefore violation of classical realism is a general feature of quantum mechanics in no way specific for entanglement.

The nonlocality, understood as a correlation beyond light cones of the systems, is a more subtle and enigmatic effect. It tacitly presumes spatially separated components in the system. This premise eventually ended up with formal identification of entangled states with nonseparable ones. The whole understanding of entanglement was formed under heavy influence of two-qubits, or more generally two-components systems, for which *Schmidt decomposition* (8) gives a transparent description and quantification of entanglement. However later on it became clear that entanglement does manifest itself in systems with no clearly separated components, e.g.

- Entanglement in an ensemble of identical bosons or fermions [35,21,20,56,14,36,63,60,68,44].

- Single particle entanglement, or entanglement of internal degrees of freedom, see [7,30] and references therein.

Nonlocality is meaningless for a condensate of identical bosons or fermions with strongly overlapping wave functions. Nevertheless we still can distinguish *coherent* Bose-Einstein condensate of bosons  $\Psi = \psi^N$  or Slater determinant for fermions  $\Psi = \psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_N$  from generic entangled states in these systems. Recall, that entangled states of biphoton were extensively studied experimentally [59,28], and Bell inequalities can be violated in such simple system as spin 1 particle, see  $n^\circ$  3.4. Thus non-locality, being indisputably the most striking manifestation of entanglement, is *not* its indispensable constituent. See also [40,41].

Lack of common ground already led to a controversy in understanding of entanglement in bosonic systems, see  $n^\circ$  3.8, and Zen question about single particle entanglement calls for a completely novel approach.

Note finally that there is no place for entanglement in von Neumann picture, where full dynamical group  $SU(\mathcal{H})$  makes all states equivalent, see  $n^\circ$  1.7. Entanglement is an effect caused by *superselection rules* or *symmetry breaking* which reduce the dynamical group to a subgroup  $G \subset SU(\mathcal{H})$  small enough to create intrinsic difference between states. For example, entanglement in two component system  $\mathcal{H}_A \otimes \mathcal{H}_B$  comes from reduction of the dynamical group to  $SU(\mathcal{H}_A) \times SU(\mathcal{H}_B) \subset SU(\mathcal{H}_A \otimes \mathcal{H}_B)$ . Therefore entanglement *must* be studied in the framework of quantum dynamical systems.

### 3.6. Completely entangled states

Roughly speaking, we consider entanglement as a manifestation of *quantum fluctuations* in a state where they come to their extreme. Specifically, we look for states with maximal total variance

$$\mathbb{D}(\psi) = \sum_i \langle \psi | X_i^2 | \psi \rangle - \langle \psi | X_i | \psi \rangle^2 = \max.$$

It follows from equation (14) that the maximum is attained for state  $\psi$  with zero expectation of all essential observables

$$\boxed{\langle \psi | X | \psi \rangle = 0, \quad \forall X \in \mathfrak{L}} \quad \begin{array}{l} \text{Entanglement} \\ \text{equation} \end{array} \quad (25)$$

We use this condition as the definition of *completely entangled* state and refer to it as *entanglement equation*. Let's outline its distinctive features.

- Equation (25) tells that in completely entangled state the system is at the center of its quantum fluctuations.
- This ensures maximality of the total variance, i.e. overall level of quantum fluctuations in the system. In this respect completely entangled states are opposite to coherent ones, and may be treated as *extremely nonclassical*. They should manifest purely quantum effects, like violation of classical realism, to the utmost.
- May be the main flaw of the conventional approach is lack of *physical quantity* associated with entanglement. In contrast to this, we consider entanglement as a manifestation of *quantum fluctuations* in a state where they come to their extreme. This, for example,

may help to understand stabilizing effect of environment on an entangled state, see [9].

- Entanglement equation (25) and the maximality of the total fluctuations plays an important heuristic rôle, similar to variational principles in mechanics. It has also a transparent geometrical meaning discussed below in  $n^\circ$  3.7. This interpretation puts entanglement in framework of Geometric Invariant Theory, which provides powerful methods for solving quantum informational problems [33].
- The total level of quantum fluctuations in irreducible system  $G : \mathcal{H}_\lambda$  varies in the range

$$\langle \lambda, 2\delta \rangle \leq \mathbb{D}(\psi) \leq \langle \lambda, \lambda + 2\delta \rangle \quad (26)$$

with minimum attained at *coherent* states, and the maximum for *completely entangled* ones, see  $n^\circ$  2.6. For spin  $s$  system this amounts to  $s \leq \mathbb{D}(\psi) \leq s(s+1)$ .

- Extremely high level of quantum fluctuations makes every completely entangled state manifestly nonclassical, see Example 3.6.2 below.
- The above definition make sense for any quantum system  $G : \mathcal{H}$  and it is in conformity with conventional one when the latter is applicable, e.g. for multi-component systems, see Example 3.6.3. For spin 1 system completely entangled spin states coincide with so called *neutrally polarized* states of biphoton, see  $n^\circ$  3.4 and [59,28].
- As expected, the definition is  $G$ -invariant, i.e. the dynamical group transforms completely entangled state  $\psi$  into completely entangled one  $g\psi$ ,  $g \in G$ .

**3.6.1 Remark.** There are few systems where completely entangled states fail to exist, e.g. in quantum system  $\mathcal{H}$  with full dynamical group  $G = \text{SU}(\mathcal{H})$  all states are coherent. In this case the total variance (12) still attains some maximum, but it doesn't satisfy entanglement equation (25). We use different terms *maximally* and *completely* entangled states to distinguish these two possibilities and to stress conceptual, rather than quantitative, origin of genuine entanglement governed by equation (25). In most cases these notions are equivalent, and all exceptions are actually known, see  $n^\circ$  3.9.

To emphasize the aforementioned difference we call quantum system  $G : \mathcal{H}$  *stable* if it contains a completely entangled state, and *unstable* otherwise.

**Example 3.6.1.** The conventional definition of entanglement explicitly refers to a *composite* system, which from our point of view is no more reasonable for entangled states, then for coherent ones. As an example let's consider completely entangled state  $\psi \in \mathcal{H}_s$  of spin  $s$  system. According to the definition this means that average spin projection onto *every* direction  $\ell$  should be zero:  $\langle \psi | J_\ell | \psi \rangle = 0$ . This certainly can't happen for  $s = 1/2$ , since in this case all states are coherent and have definite spin projection  $1/2$  onto some direction. But for  $s \geq 1$  such states do exist and will be described later in  $n^\circ$  3.11. For example, one can take  $\psi = |0\rangle$  for integral spin  $s$ , and

$$\psi = \frac{1}{\sqrt{2}}(|+s\rangle - |-s\rangle)$$

for any  $s \geq 1$ . They have extremely big fluctuations  $\mathbb{D}(\psi) = s(s+1)$ , and therefor are *manifestly nonclassical*: average spin projection onto every direction is zero, while the standard deviation  $\sqrt{s(s+1)}$  exceeds maximum of the spin projection  $s$ .

**Example 3.6.2.** This consideration can be literally transferred to an arbitrary irreducible system  $G : \mathcal{H}_\lambda$ , using inequality  $\langle \lambda, \lambda \rangle < \langle \lambda, \lambda + 2\delta \rangle$  instead of  $s^2 < s(s+1)$ , to the effect that a completely entangled state of any system is nonclassical.

**Example 3.6.3.** Entanglement equation (25) implies that state of a multicomponent system, say  $\psi \in \mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ , is completely entangled iff its marginals  $\rho_A, \rho_B, \rho_C$  are scalar operators. This observation is in conformity with conventional approach to entanglement [13], cf. also Example 1.6.1.

### 3.7. General entangled states and stability

From operational point of view state  $\psi \in \mathcal{H}$  is *entangled* iff one can filter out from  $\psi$  a completely entangled state  $\psi_0$  using SLOCC operations. As we know from Example 2.4.2 in standard quantum information settings SLOCC group coincide with complexification  $G^c$  of the dynamic group  $G$ . This leads us to the following

**Definition 3.7.1.** State  $\psi \in \mathcal{H}$  of dynamical system  $G : \mathcal{H}$  is said to be *entangled* iff it can be transformed into a completely entangled state  $\psi_0 = g\psi$  by complexified group  $G^c$  (possibly asymptotically  $\psi_0 = \lim_i g_i \psi$  for some sequence  $g_i \in G^c$ ).

In Geometric Invariant Theory such states  $\psi$  are called *stable* (or *semistable* if  $\psi_0$  can be reached only asymptotically). Their intrinsic characterization is one of the central problems both in Invariant Theory and in Quantum Information. Relation between these two theories can be summarized in the following table, with some entries to be explained below.

DICTIONARY

Quantum Information	Invariant Theory
Entangled state	Semistable vector
Disentangled state	Unstable vector
SLOCC transform	Action of complexified group $G^c$
Completely entangled state $\psi_0$ prepared from $\psi$ by SLOCC	Minimal vector $\psi_0$ in complex orbit of $\psi$
State obtained from completely entangled one by SLOCC	Stable vector

Completely entangled states can be characterized by the following theorem, known as *Kempf–Ness unitary trick*.

**Theorem 3.7.2** (Kempf-Ness [29]). *State  $\psi \in \mathcal{H}$  is completely entangled iff it has minimal length in its complex orbit*

$$|\psi| \leq |g \cdot \psi|, \quad \forall g \in G^c. \quad (27)$$

*Complex orbit  $G^c\psi$  contains a completely entangled state iff it is closed. In this case the completely entangled state is unique up to action of  $G$ .*

**3.7.3 Remark.** Recall that entangled state  $\psi$  can be *asymptotically* transformed by SLOCC into a completely entangled one. By Kempf-Ness theorem the question when this can be done *effectively* depends on whether the complex orbit of  $\psi$  is closed or not. The following result gives a necessary condition for this.

**Theorem** (Matsushima [42]). *Complex stabilizer  $(G^c)_\psi$  of stable state  $\psi$  coincides with complexification of its compact stabilizer  $(G_\psi)^c$ .*

Square of length of the minimal vector in complex orbit

$$\mu(\psi) = \inf_{g \in G^c} |g\psi|^2, \quad (28)$$

provides a natural quantification of entanglement. It amounts to  $\cos 2\varphi$  for spin 1 state (20), to *concurrence*  $C(\psi)$  [26] in two qubits, and to square root of *3-tangle*  $\tau(\psi)$  for three qubits (see below). We call it *generalized concurrence*. Evidently  $0 \leq \mu(\psi) \leq 1$ .

Equation  $\mu(\psi) = 1$  tells that  $\psi$  is already a minimal vector, hence completely entangled state.

Nonvanishing of the generalized concurrence  $\mu(\psi) > 0$  means that closure of complex orbit  $\overline{G^c\psi}$  doesn't contain zero. Then the orbit of minimal dimension  $\mathcal{O} \subset \overline{G^c\psi}$  is closed and nonzero. Hence by Kempf-Ness unitary trick it contains a completely entangled state  $\psi_0 \in \mathcal{O}$  which asymptotically can be obtained from  $\psi$  by action of the complexified dynamical group. Therefore by definition 3.7.1

$$\mu(\psi) > 0 \iff \psi \text{ is entangled.}$$

### 3.8. Coherent versus unstable states

The minimal value  $\mu(\psi) = 0$  corresponds to *unstable* vectors that can asymptotically fall into zero under action of the complexified dynamical group. They form so-called *null cone*. It contains all coherent states, along with some others degenerate states, like  $W$ -state in three qubits, see Example 3.10.1.

Noncoherent unstable states cause many controversies. There is unanimous agreement that coherent states are disentangled. In approach pursued in [63] all noncoherent states are treated as entangled. Other researchers [21,20] argue that some noncoherent unstable bosonic states are actually disentangled. From our operational point of view all unstable states should be treated as disentangled, since they can't be filtered out into a completely entangled state even asymptotically. Therefore we accept the equivalence

$$\text{DISENTANGLED} \iff \text{UNSTABLE} \iff \text{NOT SEMISTABLE.}$$

### 3.8.1. Systems in which all unstable states are coherent

The above controversy vanishes iff the null cone contains only coherent states, or equivalently dynamical group  $G$  acts transitively on unstable states. Spin one and two qubits systems are the most notorious examples. They are low dimensional *orthogonal systems* with dynamical group  $\mathrm{SO}(n)$  acting in  $\mathcal{H}^n = \mathbb{E}^n \otimes \mathbb{C}$  by Euclidean rotations. Null cone in this case consists of isotropic vectors  $(x, x) = 0$ , which are at the same time coherent states, cf.  $n^\circ$  3.4.

**Theorem 3.8.1.** *Stable irreducible system  $G : \mathcal{H}$  in which all unstable states are coherent is one of the following*

- *Orthogonal system  $\mathrm{SO}(\mathcal{H}) : \mathcal{H}$ ,*
- *Spinor representation of group  $\mathrm{Spin}(7)$  of dimension 8,*
- *Exceptional group  $G_2$  in its fundamental representation of dimension 7.*

The theorem can be deduced from Theorem 2.7.1 characterizing coherent states by quadratic equations. Indeed, the null cone is given by vanishing of all invariants. Hence in conditions of the theorem the fundamental invariants should have degree two. For irreducible representation there is at most one invariant of degree two, the invariant metric  $(x, y)$ . Thus the problem reduces to description of subgroups  $G \subset \mathrm{SO}(\mathcal{H})$  acting transitively on isotropic cone  $(x, x) = 0$ . The metric  $(x, x)$  is unique basic invariant of such system. Looking into the table in Vinberg-Popov book [62] we find only one indecomposable system with unique basic invariant of degree two not listed in the theorem: spinor representation of  $\mathrm{Spin}(9)$  of dimension 16 studied by Igusa [27]. However, as we'll see below, the action of this group  $\mathrm{Spin}(9)$  on the isotropic cone is not transitive.

Coherent states of decomposable irreducible system  $G_A \times G_B : \mathcal{H}_A \otimes \mathcal{H}_B$  are products  $\psi_A \otimes \psi_B$  of coherent states of the components. Hence codimension of the cone of coherent states is at least  $d_A d_B - d_A - d_B + 1 = (d_A - 1)(d_B - 1)$ . As we've seen above, in conditions of the theorem the codimension should be equal to one, which is possible only for system of two qubits  $d_A = d_B = 2$ , which is equivalent to orthogonal system of dimension four. One can also argue that projective quadric  $Q : (x, x) = 0$  of dimension greater than two is indecomposable  $Q \neq X \times Y$ .  $\square$

Both exceptional systems carry an invariant symmetric form  $(x, y)$ . Scalar square  $(x, x)$  generates the algebra of invariants, and therefore the null cone consists of isotropic vectors  $(x, x) = 0$ , as in the orthogonal case. These mysterious systems emerge also as exceptional *holonomy groups* of Riemann manifolds [2]. Their physical meaning is unclear.

Élie Cartan [8] carefully studied coherent states in irreducible (half)spinor representations of  $\mathrm{Spin}(n)$  of dimension  $2^\nu$ ,  $\nu = \lfloor \frac{n-1}{2} \rfloor$ . He call them *pure spinors*. In general the cone of pure spinors is intersection of  $2^{\nu-1}(2^\nu + 1) - \binom{2^\nu+1}{\nu}$  linear independent quadrics.

For  $n < 7$  there are no equations, i.e. all states are coherent. In such systems there is no entanglement whatsoever, and we exclude them from the theorem. These systems are very special and have a transparent physical interpretation.

- For  $n = 3$  spinor representation of dimension two identifies  $\mathrm{Spin}(3)$  with  $\mathrm{SU}(2)$ . Vector representation of  $\mathrm{SO}(3)$  is just spin 1 system, studied in  $n^\circ$  3.4.
- Two dimensional halfspinor representations of  $\mathrm{Spin}(4)$  identify this group with  $\mathrm{SU}(2) \times \mathrm{SU}(2)$  and the orthogonal system of dimension 4 with two qubits.
- For  $n = 5$  spinor representation  $\mathcal{H}^4$  of dimension 4 carries invariant symplectic form  $\omega$  and identify  $\mathrm{Spin}(5)$  with symplectic group  $\mathrm{Sp}(\mathcal{H}^4, \omega)$ . The standard vector representation of  $\mathrm{SO}(5)$  in this settings can be identified with the space of skew symmetric forms in  $\mathcal{H}^4$  modulo the defining form  $\omega$ .

- For  $n = 6$  halfspinor representations of dimension 4 identify  $\text{Spin}(6)$  with  $\text{SU}(\mathcal{H}^4)$  and the orthogonal system of dimension 6 with  $\text{SU}(\mathcal{H}^4) : \wedge^2 \mathcal{H}^4$ . This is a system of two fermions of rank 4. The previous group  $\text{Spin}(5) \simeq \text{Sp}(\mathcal{H}^4)$  is just a stabilizer of a generic state  $\omega \in \wedge^2 \mathcal{H}^4$ .

In the next case  $n = 7$  coherent states are defined by single equation  $(x, x) = 0$  and coincide with unstable ones. Thus we arrive at the first special system  $\text{Spin}(7) : \mathcal{H}^8$ .

Stabilizer of a non isotropic spinor  $\psi \in \mathcal{H}^8$ ,  $(\psi, \psi) \neq 0$  in  $\text{Spin}(7)$  is exceptional group  $G_2$  and its representation in orthogonal complement to  $\psi$  gives the second system  $G_2 : \mathcal{H}^7$ . Alternatively it can be described as representation of automorphism group of Cayley octonic algebra in the space of purely imaginary octaves.

Halfspinor representations of  $\text{Spin}(8) : \mathcal{H}^8$  also carry invariant symmetric form  $(x, y)$ . It follows that  $\text{Spin}(8)$  acts on halfspinors as *full* group of orthogonal transformations. Hence these representations are geometrically equivalent to the orthogonal system  $\text{SO}(\mathcal{H}^8) : \mathcal{H}^8$ . The equivalence is known as Cartan's *triality* [8].

Finally spinor representation of  $\text{Spin}(9)$  of dimension 16 also carries invariant symmetric form  $(x, y)$  which is unique basic invariant of this representation. However according to Cartan's formula the cone of pure spinors is intersection of 10 independent quadrics, hence differs from the null cone  $(x, x) = 0$ .

### 3.8.2. Fermionic realization of spinor representations

Spinor representations of two fold covering  $\text{Spin}(2n)$  of orthogonal group  $\text{SO}(2n)$  have a natural *physical realization*. Recall that all *quadratic* expressions in creation and annihilation operators  $a_i^\dagger, a_j$ ,  $i, j = 1 \dots n$  in a system of fermions with  $n$  intrinsic degrees of freedom form orthogonal Lie algebras  $\mathfrak{so}(2n)$  augmented by scalar operator (to avoid scalars one have to use  $\frac{1}{2}(a_i^\dagger a_i - a_i a_i^\dagger)$  instead of  $a_i^\dagger a_i, a_i a_i^\dagger$ ). It acts in fermionic Fock space  $\mathbb{F}(n)$ , known as *spinor representation* of  $\mathfrak{so}(2n)$ . In difference with bosonic case it has finite dimension  $\dim \mathbb{F}(n) = 2^n$  and splits into two *halfspinor* irreducible components  $\mathbb{F}(n) = \mathbb{F}_{ev}(n) \oplus \mathbb{F}_{odd}(n)$ , containing even and odd number of fermions respectively.

For fermions of dimension  $n = 4$  the halfspinors can be transformed into vectors by the Cartan's triality. This provides a physical interpretation of the orthogonal system of dimension 8.

To sum up, orthogonal systems of dimension  $n = 3, 4, 6, 8$  have the following physical description

- $n = 3$ . Spin 1 system.
- $n = 4$ . Two qubit system.
- $n = 6$ . System of two fermions  $\text{SU}(\mathcal{H}^4) : \wedge^2 \mathcal{H}^4$  of dimension 4.
- $n = 8$ . System of fermions of dimension 4 with variable number of particles (either even or odd).

The last example is fermionic analogue of a system of quantum oscillators  $n^\circ 2.1$ . Lack of the aforementioned controversy makes description of pure and mixed entanglement in orthogonal systems very transparent, and quite similar to that of two qubit and spin 1 systems, see  $n^\circ ??$ .



### 3.9. Unstable systems

Halfspinor representations of the next group  $\text{Spin}(10)$  was discussed as an intriguing possibility, that quarks and leptons may be composed of five different species of fundamental fermionic objects [69,66]. This is a very special system where all states are unstable, hence disentangled. In other words the null cone amounts to the whole state space and there is no genuine entanglement governed by equation (25). Such systems are opposite to those considered in the preceding section, where the null cone is as small as possible. We call them *unstable*. There are very few types of such indecomposable irreducible dynamical systems [62,43]:

- Unitary system  $\text{SU}(\mathcal{H}) : \mathcal{H}$ ;
- Symplectic system  $\text{Sp}(\mathcal{H}) : \mathcal{H}$ ;
- System of two fermions  $\text{SU}(\mathcal{H}) : \wedge^2 \mathcal{H}$  of odd dimension  $\dim \mathcal{H} = 2k + 1$ ;
- A halfspinor representation of dimension 16 of  $\text{Spin}(10)$ .

All (half)spinor irreducible representations for  $n < 7$  fall into this category. There are many more such composite systems, and their classification is also known due to M. Sato and T. Kimura [55].

### 3.10. Classical criterion of entanglement

Kempf–Ness theorem 3.7.2 identifies closed orbits of complexified group  $G^c$  with completely entangled states modulo action of  $G$ . Closed orbits can be separated by  $G$ -invariant polynomials. This leads to the following *classical criterion* of entanglement.

**Theorem 3.10.1** (Classical Criterion). *State  $\psi \in \mathcal{H}$  is entangled iff it can be separated from zero by  $G$ -invariant polynomial*

$$f(\psi) \neq f(0), \quad f(gx) = f(x), \forall g \in G, x \in \mathcal{H}.$$

**Example 3.10.1.** For two component system  $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$  all invariants are polynomials in  $\det[\psi_{ij}]$  (no invariants for  $\dim \mathcal{H}_A \neq \dim \mathcal{H}_B$ ). Hence state is entangled iff  $\det[\psi_{ij}] \neq 0$ . The generalized concurrence (28) related to this basic invariant by equation

$$\mu(\psi) = n |\det[\psi_{ij}]|^{2/n}.$$

Unique basic invariant for 3-qubit is *Cayley hyperdeterminant* [?,18]

$$\begin{aligned} \text{Det}[\psi] = & (\psi_{000}^2 \psi_{111}^2 + \psi_{001}^2 \psi_{110}^2 + \psi_{010}^2 \psi_{101}^2 + \psi_{011}^2 \psi_{100}^2) \\ & - 2(\psi_{000} \psi_{001} \psi_{110} \psi_{111} + \psi_{000} \psi_{010} \psi_{101} \psi_{111} \\ & + \psi_{000} \psi_{011} \psi_{100} \psi_{111} + \psi_{001} \psi_{010} \psi_{101} \psi_{110} \\ & + \psi_{001} \psi_{011} \psi_{110} \psi_{100} + \psi_{010} \psi_{011} \psi_{101} \psi_{100}) \\ & + 4(\psi_{000} \psi_{011} \psi_{101} \psi_{110} + \psi_{001} \psi_{010} \psi_{100} \psi_{111}). \end{aligned}$$

related to 3-tangle [10] and generalized concurrence (28) by equations

$$\tau(\psi) = 4|\text{Det}[\psi]|, \quad \mu(\psi) = \sqrt{\tau(\psi)}.$$

One can check that Cayley hyperdeterminant vanishes for so called *W-state*

$$W = \frac{|100\rangle + |010\rangle + |001\rangle}{\sqrt{3}}$$

which therefor is neither entangled nor coherent.

**3.10.2 Remark.** This examples elucidate the nature of entanglement introduced here. It takes into account only those entangled states that spread over the whole system, and disregards any entanglement supported in a smaller subsystem, very much like 3-tangle did. For example, absence of entanglement in two component system  $\mathcal{H}_A \otimes \mathcal{H}_B$  for  $\dim \mathcal{H}_A \neq \dim \mathcal{H}_B$  reflects the fact that in this case every state belongs to a smaller subspace  $V_A \otimes V_B$ ,  $V_A \subset \mathcal{H}_A$ ,  $V_B \subset \mathcal{H}_B$  as it follows from Schmidt decomposition (8). Entanglement of such states should be treated in the corresponding subsystems.

### 3.11. Hilbert-Mumford criterion

The above examples, based on Theorem 3.10.1, shows that invariants are essential for understanding and quantifying of entanglement. Unfortunately finding invariants is a tough job, and more then 100 years of study give no hope for a simple solution.

There are few cases where all invariants are known, some of them were mentioned above. In addition invariants and covariants of four qubits and three qutrits were found recently [39,4,5]. For five qubit only partial results are available [38]. See more on invariants of qubits in [45,46]. For system of format  $4 \times 4 \times 2$  the invariants are given in [51].

Spin systems have an equivalent description in terms of *binary forms*, see Example 3.11.2. Their invariants are described by theory of *Binary Quantics*, diligently pursued by mathematicians from the second half of 19-th century. This is an amazingly difficult job, and complete success was achieved by classics for  $s \leq 3$ , the cases  $s = 5/2$  and 3 being one of the crowning glories of the theory [43]. Modern authors advanced it up to  $s = 4$ .

Other classical results of invariant theory are still waiting physical interpretation and applications. In a broader context Bryce S. DeWitt described the situation as follows:

*“Why should we not go directly to invariants? The whole of physics is contained in them. The answer is that it would be fine if we could do it. But it is not easy.”*

Now, due to Hilbert’s insight, we know that the difficulty is rooted in a perverse desire to put geometry into Procrustean bed of algebra. He created *Geometric Invariant Theory* just to overcome it.

**Theorem 3.11.1** (Hilbert-Mumford Criterion [43]). *State  $\psi \in \mathcal{H}$  is entangled iff every observable  $X \in \mathfrak{L} = \text{Lie}(G)$  of the system in state  $\psi$  assumes a nonnegative value with positive probability.*

By changing  $X$  to  $-X$  one deduces that  $X$  should assume nonpositive values as well. So in entangled state no observable can be biased neither to strictly positive nor to strictly negative values. Evidently completely entangled states with zero expectations  $\langle \psi | X | \psi \rangle = 0$  of all observables pass this test.

**Example 3.11.1.** Let  $X = X_A \otimes 1 + 1 \otimes X_B$  be observable of two qubit system  $\mathcal{H}_A \otimes \mathcal{H}_B$  with

$$\text{Spec } X_A = \pm\alpha, \quad \text{Spec } X_B = \pm\beta, \quad \alpha \geq \beta \geq 0.$$

Suppose that  $\psi$  is *unstable* and observable  $X$  assumes only strictly positive values in state  $\psi$ . Since those values are  $\alpha \pm \beta$  then the state is decomposable

$$\psi = a|\alpha\rangle \otimes |\beta\rangle + b|\alpha\rangle \otimes |-\beta\rangle = |\alpha\rangle \otimes (a|\beta\rangle + b|-\beta\rangle),$$

i.e. Hilbert-Mumford criterion characterizes entangled qubits.

The general form of H-M criterion may shed some light on the nature of entanglement. However, it was originally designed for application to *geometrical objects*, like linear subspaces or algebraic varieties of higher degree, and its efficacy entirely depends on our ability to express it in geometrical terms. Let's give an example.

**Example 3.11.2.** *Stability of spin states.* Spin  $s$  representation  $\mathcal{H}_s$  can be realized in space of *binary forms*  $f(x, y)$  of degree  $d = 2s$

$$\mathcal{H}_s = \{f(x, y) \mid \deg f = 2s\}$$

in which  $\text{SU}(2)$  acts by linear substitutions  $f(x, y) \mapsto f(ax + by, cx + dy)$ . To make swap from physics to mathematics easier we denote by  $f_\psi(x, y)$  the form corresponding to state  $\psi \in \mathcal{H}_s$ . Spin state  $\psi \in \mathcal{H}_s$  can be treated algebraically, physically, or geometrically according to the following equations

$$\psi = \sum_{\mu=-s}^{\mu=s} a_\mu \binom{2s}{s+\mu} x^{s+\mu} y^{s-\mu} = \sum_{\mu=-s}^{\mu=s} a_\mu \binom{2s}{s+\mu}^{1/2} |\mu\rangle = \prod_i (\alpha_i x - \beta_i y).$$

The first one is purely algebraic, the second gives physical decomposition over eigenstates

$$|\mu\rangle = \binom{2s}{s+\mu}^{1/2} x^{s+\mu} y^{s-\mu}, \quad J_z |\mu\rangle = \mu |\mu\rangle$$

of spin projector operator  $J_z = \frac{1}{2} \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$ , and the last one is geometrical. It describes form  $f_\psi(x, y)$  in terms of configuration of its roots  $z_i = (\beta_i : \alpha_i)$  in *Riemann sphere*  $\mathbb{C} \cup \infty = \mathbb{S}^2$  (known also as *Bloch sphere* for spin  $1/2$  states, and *Poincaré sphere* for polarization of light).

According to H-M criterion state  $\psi$  is *unstable* iff spin projections onto some direction  $\ell$  are strictly positive. By rotation we reduce the problem to  $z$ -component  $J_z = \frac{1}{2} \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$  in which case the corresponding form

$$f_\psi(x, y) = \sum_{\mu>0} a_\mu \binom{2s}{s+\mu}^{1/2} |\mu\rangle = \sum_{\mu>0} a_\mu \binom{2s}{s+\mu} x^{s+\mu} y^{s-\mu}$$

has root  $x = 0$  of multiplicity more then  $s = d/2$ . As result we arrive at the following criterion of entanglement (=semistability) for spin states

$$\psi \text{ is entangled} \iff \text{no more then half of the roots } f_\psi(x, y) \text{ coincide.} \quad (29)$$

One can show that if *less* then half of the roots coincide then the state is *stable* i.e. can be transformed into a completely entangled one by Lorentz group  $SL(2, \mathbb{C})$  acting on roots  $z_i \in \mathbb{C} \cup \infty$  by Möbius transformations  $z \mapsto \frac{az+b}{cz+d}$ . In terms of these roots entanglement equation (25) amounts to the following condition

$$\psi \text{ completely entangled} \iff \sum_i (z_i) = 0, \quad (30)$$

where parentheses denote *unit vector*  $(z_i) \in \mathbb{S}^2 \subset \mathbb{E}^3$  mapping into  $z_i \in \mathbb{C} \cup \infty$  under stereographic projection. For example, for integral spin completely entangled state  $|0\rangle$  can be obtained by putting equal number of points at the North and the South poles of Riemann sphere. Another balanced configuration (30) consisting of  $2s$  points evenly distributed along the equator produces completely entangled state  $|\psi\rangle = \frac{1}{\sqrt{2}}(|s\rangle - |-s\rangle)$  for any  $s \geq 1$ , cf. Example 3.6.1.

Note also that a configuration with half of its points in the South pole can't be transformed into a balanced one (30), except all the remaining points are at the North. However this can be done asymptotically by homothety  $z \mapsto \lambda z$  as  $\lambda \rightarrow \infty$  which sends all points except zero to infinity. This gives an example of *semistable* but *not* stable configuration.

*Summary.* Solvability of the *nonlinear problem* of conformal transformation of a given configuration into a balanced one (30) depends on *topological condition* (29) on its multiplicities. One can find application of this principle to quantum marginal problem in [33,34].

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